ON THE STABILITY OF PROPAGATION OF SPHERICAL FLAMES

A. G. Istratov and V. B. Librovich

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The stability of propagation of a plane flame in the quasi-steady regime has been investigated by L. D. Landau [1, 2], who treated the laminar flame as a discontinuity of density, temperature, pressure, and velocity propagating through the gas at a constant speed. The result was unexpected: the flame was predicted to be absolutely unstable. The disagreement between this theoretical prediction and known experimental data has led to several theoretical and experimental studies [3–8].

The theoretical studies (reviewed in [4]) considered the stabilizing effect of transport processes (viscosity, thermal conductivity, diffusion) for the case in which the wavelength of the perturbations was of the order of the flame front thickness. In this case the flame front cannot be treated as an infinitesimally thin hydrodynamic discontinuity, as in Landau's work, and the effect of the perturbations on the structure and speed of the flame front must be taken into account. The best known of these studies is that of Markstein [3], who accounted for the transfer processes by introducing a constant which related the change of the speed of the flame front to its curvature. (It should be noted that long before Markstein's work [3], Zel'dovich [9] had noted the effect of the ratio between the thermal and mass diffusivities of the component which controls the chemical reaction on the stability of the flame.)

The studies reported in [3, 4] did not explain the disagreement between Landau's theory and experiment. The critical Reynolds numbers characterizing the onset of instability which were obtained in these studies were of order unity, whereas experimentally one finds stable flames at much higher Reynolds numbers.

The experimental studies reported in [5–8] attempted to observe the instability predicted by Landau and to measure the critical conditions. In view of the fact that experiments involving flame propagation in tubes or combustion in various burners are complicated by secondary effects (turbulence, heat transfer, etc.) which are neglected in the theory, it is more convenient to study a spherical flame propagating through a combustible gas. Such experiments, carried out first by Zel'dovich and Rozlovskii [5], and later by Shchelkin, Troshchin, and their collaborators [6–8], showed that the critical Reynolds numbers are of order 10^4–10^6.

However, these studies neglected the fact that in the case of a spherical flame the instability is affected by the increase of flame front surface area. In particular, the instability must be defined in a different way than in the plane case, since a deformation of the flame front and a change of its velocity of propagation can be observed only if the amplitude of the perturbation increases faster than the dimensions of the flame. In this paper the criterion for instability is the increase with time of the relative amplitude of perturbation of the flame surface (the ratio of the amplitude and the instantaneous radius). Our analysis of the stability of propagation of a spherical flame is based on the hydrodynamic formulation of Landau and Markstein. We show that the flame is stable with respect to the low spherical harmonics of the perturbations, because the rate of growth of these harmonics is lower than the speed of propagation of the flame. As regards higher harmonics, we find that the curvature of the perturbations has a significant effect on the speed of propagation of the flame (this was taken into account in Markstein's approximation [3]), which stabilizes the flame up to a certain point in time. This time determines the critical Reynolds number.

It will be shown that the critical Reynolds number can reach a relatively high value, 10^3–10^4, if one interprets Markstein's constant as a parameter strongly dependent on the activation energy of the chemical reaction in the flame. This interpretation gives better agreement with experimental results [5–8]. The theoretical model is also in good qualitative agreement with experimental observations of the development of the perturbations.

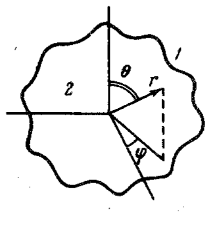

Fig. 1

NOTATION

r, θ, φ—spherical coordinates, t—time, u_n—normal speed of propagation of the flame, α—ratio of densities of hot and cold gas, ξ, θ, φ—dimensionless spherical coordinates, τ—dimensionless time, v_r, v_θ, v_φ—velocity components in spherical coordinates, p—pressure, ρ—density, ψ—velocity potential, Φ—source strength, R—flame radius, A—perturbation amplitude, λ—perturbation wavelength, k—wave number, ε—perturbation of flame surface, ζ—dimensionless perturbation of flame surface, Λ—radius of curvature, n—index of spherical harmonic, m—periodicity parameter with respect to the angle φ, f_i—unknown functions ($i = 1, 2, 3, 4$), ω_i—roots of characteristic equation for the spherical flame, Ω_i—roots of the characteristic equation for the plane flame, μ—Markstein's constant,

$$v_2 = \frac{u_n}{\alpha}, \qquad \xi = \frac{r}{v_2 t}, \qquad \tau = \ln t,$$

$$R = v_2 t, \qquad \zeta = \frac{\varepsilon}{v_2 t}, \qquad k = \frac{2\pi}{\lambda}.$$

Superscripts: °—unperturbed variable; ′—perturbation. Subscripts: 1, 2—cold and hot gas, respectively; 0—infinity ($r \to \infty$).

§1. Formulation of the problem.

Our analysis of the stability of a spherical flame propagating through a space filled with combustible gas will be based on the assumptions of Landau and Markstein [1–3]. We shall regard the flame front as a surface of discontinuity of velocity, density, temperature, and pressure which propagates through the gas with a speed which depends, in general, on the curvature of the surface. We assume that the flame propagates through an ideal incompressible gas (the density is independent of pressure, but has different values on the two sides of the flame front); this assumption is justified if the flame speed is much lower than the speed of sound.

The velocity and pressure perturbations are governed by the linearized continuity and Euler equations

$$\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}^\circ \nabla)\mathbf{v}' = -\frac{1}{\rho}\nabla p', \tag{1.1}$$

$$\operatorname{div} \mathbf{v}' = 0. \tag{1.2}$$

These equations apply both to the cold combustible gas and to the hot combustion products—regions 1 and 2, respectively, in Fig. 1, which shows the perturbed spherical flame. At the flame front the perturbations must satisfy conservation of mass and momentum, and also the relation between the flame speed and the perturbations.

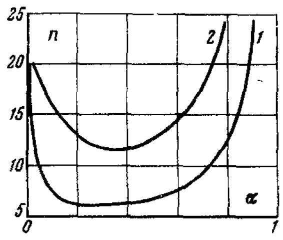

Fig. 2

To solve this problem, it is convenient to use spherical coordinates (Fig. 1) and to expand all perturbations in a series in terms of a complete orthogonal system of noninteracting spherical harmonics. For the flame to be unstable it is necessary that the amplitude of perturbation divided by the instantaneous radius of the flame grow with time at least for one spherical harmonic. Therefore, without loss of generality, we may investigate the stability of the flame with respect to a perturbation in the form of a single spherical harmonic.

In order to simplify the analysis, we shall solve the problem in two stages. First, we shall investigate the stability of the flame with respect to low spherical harmonics, in which case it may be assumed that the speed of propagation is independent of the perturbations (Landau's assumption). Subsequently we shall investigate the case of higher harmonics, using Markstein's relation between the flame speed and the perturbations [3].

§2. Hydrodynamic formulation of the unperturbed propagation of a spherical flame. A spherical flame, which causes the thermal expansion of the gas, has the same effect as a gas source. This analogy can be used to determine the velocity and pressure distribution ahead of the unperturbed flame (in region 1). The velocity potential in source flow is

$$\psi_1^\circ = -\frac{\Phi}{4\pi r} \qquad (\Phi - \text{source strength}). \qquad (2.1)$$

The equivalent source strength of the flame can be calculated from the change of volume of the gas passing through the flame front in unit time,

$$\Phi = 4\pi R^2 u_n \left(\frac{1}{\alpha} - 1\right) = 4\pi v_2^3 t^2 (1-\alpha). \qquad (2.2)$$

Consequently, the radial velocity of the cold gas is

$$v_{r1}^\circ = \frac{\partial \psi_1^\circ}{\partial r} = (1-\alpha) v_2^3 \frac{t^2}{r^2}. \qquad (2.3)$$

The pressure distribution in region 1 can be found by means of the Lagrange-Cauchy integral. This yields

$$\frac{p_1^\circ - p_0}{\rho_1 v_2^2} = 2(1-\alpha)\frac{v_2 t}{r}\left[1 - \frac{(1-\alpha) v_2^3}{4}\frac{t^3}{r^3}\right]. \qquad (2.4)$$

In region 2, which contains hot combustion products, the gas comes to rest ($v_{r2}^\circ = 0$) and the pressure is constant and equal to

$$p_2^\circ = p_0 + \rho_1 v_2^2 \frac{(1-\alpha)(3-\alpha)}{2}. \qquad (2.5)$$

(Equation (2.5) can be obtained from (2.4) and the condition for the pressure discontinuity at the front.)

§3. Stability analysis with respect to low spherical harmonics. Following Landau, we shall assume that the flame speed is constant. This solution will be applicable to large-scale perturbations, for which one may neglect their effect on the structure of the flame.

It should be noted that in the case of a spherical flame one should expect a power-law dependence of the perturbations on time, rather than an exponential dependence as in the case of a plane flame. This follows, as has been shown by Zel'dovich, from the following simple argument: From dimensional considerations it can be shown that the rate of change of the amplitude of the perturbations dA/dt must satisfy the relation

$$\frac{dA}{dt} \sim \frac{A u_n}{\lambda}. \qquad (3.1)$$

In the case of a plane flame the wavelength of the perturbation is λ = const, and (3.1) then yields an exponential relation. In the case of a spherical flame the wavelength corresponding to a given spherical harmonic grows linearly with time, and (3.1) then yields a power-law dependence of A on time.

To solve the problem, we must specify the linearized boundary conditions at the flame front. Unlike the boundary conditions in [1], in the present case the unperturbed variables v_r°, p° are functions of the radial coordinate r and the time t. Thus, at $r = R(t)$ we have: conservation of mass

$$v_{r1}' - \frac{\partial \varepsilon}{\partial t} + \frac{\partial v_{r1}^\circ}{\partial r}\varepsilon = \alpha\left(v_{r2}' - \frac{\partial \varepsilon}{\partial t}\right); \qquad (3.2)$$

conservation of normal momentum

$$p_1' + \frac{\partial p_1^\circ}{\partial r}\varepsilon = p_2'; \qquad (3.3)$$

conservation of tangential momentum (in the φ = const plane)

$$v_{\theta 1}' + \frac{v_{r1}^\circ}{R}\frac{\partial \varepsilon}{\partial \theta} = v_{\theta 2}'; \qquad (3.4)$$

conservation of tangential momentum (in the θ = const plane)

$$v_{\varphi_1}' + \frac{v_{r1}^\circ}{R\sin\theta}\frac{\partial \varepsilon}{\partial \varphi} = v_{\varphi 2}'; \qquad (3.5)$$

constant normal speed of propagation of the flame

$$v_{r1}' + \frac{\partial v_{r1}^\circ}{\partial r}\varepsilon - \frac{\partial \varepsilon}{\partial t} = 0. \qquad (3.6)$$

In addition to these conditions at the flame front, the solution must satisfy the following obvious conditions for the absence of perturbations far away from the flame and for the boundedness of the perturbations at the center:

$$v_{r1}',\ v_{\theta 1}',\ v_{\varphi 1}',\ p_1' \to 0 \quad \text{as} \quad r \to \infty$$
$$|v_{r2}'|,\ |v_{\theta 2}'|,\ |v_{\varphi 2}'|,\ |p_2'| < \infty \quad \text{as} \quad r \to 0. \tag{3.7}$$

Solving (1.1) and (1.2), we shall use different methods in regions 1 and 2. In region 1, containing the combustible gas, we shall use the condition that the perturbed flow is a potential flow. This does not restrict the generality of the solution—the perturbations which appear in the gas are determined by the properties of the flame surface and the particles in region 1 are not directly affected by the perturbing forces. Thus the potential nature of the flow is preserved (Helmholtz's theorem).

In region 1 the velocity potential ψ_1' satisfies Laplace's equation

$$\Delta\psi_1' = 0. \tag{3.8}$$

To solve Eq. (3.8) it is convenient to replace the coordinates r, θ, φ, t by the dimensionless coordinates $\xi = r/v_2 t$, θ, φ, $\tau = \ln t$. The introduction of the coordinates ξ, τ is natural, in view of the fact that the propagation of the unperturbed flame is self-similar.

The solution of Laplace's equation in spherical coordinates can be represented by a series of terms of the form

$$\psi_{1,1}' = \Psi_{mn}^{1}(\tau)\sin m\varphi\, P_n^m(\cos\theta) / \xi^{n+1},$$
$$\psi_{1,2}' = \Psi_{mn}^{2}(\tau)\cos m\varphi\, P_n^m(\cos\theta) / \xi^{n+1}, \tag{3.9}$$

where m and n are integers and m < n. As we said in §1, we can restrict our attention to the terms (3.9) corresponding to a single value of m and n. It is also sufficient to consider the term containing sin mφ only, since the result for the cosine term is identical.

Introducing for convenience the new function $f_1(\tau)$, defined by

$$\Psi_{mn}^{1}(\tau) = v_2^2 e^{\tau} f_1(\tau), \tag{3.10}$$

we obtain the following expressions for the components of the perturbed velocity in region 1:

$$v_{\xi 1}' / v_2 = -(n+1) f_1(\tau)\, \xi^{(-n+2)} \sin m\varphi\, P_n^m(\cos\theta), \tag{3.11}$$

$$v_{\theta 1}' / v_2 = f_1(\tau)\, \xi^{-(n+2)} \sin m\varphi\, dP_n^m / d\theta, \tag{3.12}$$

$$v_{\varphi 1}' / v_2 = f_1(\tau)\, m \xi^{-(n+2)} \cos m\varphi\, P_n^m(\cos\theta) / \sin\theta. \tag{3.13}$$

From the linearized Cauchy-Lagrange integral,

$$\frac{p_1'}{\rho_1} = -\frac{\partial\psi_1'}{\partial t} - v_{r1}{}^{\circ} v_{r1}' =$$
$$= -e^{-\tau}\frac{\partial\psi_1'}{\partial\tau} + \xi e^{-\tau}\frac{\partial\psi_1'}{\partial\xi} - (1-\alpha)\frac{v_2 v_{\xi 1}'}{\xi^2}, \tag{3.14}$$

we obtain the pressure perturbation

$$\frac{p_1'}{\rho_1 v_2^2} = -\xi^{-(n+1)} \sin m\varphi P_n^m \Big[(n+2) f_1 -$$
$$- \frac{(1-\alpha)(n+1)}{\xi^3} f_1 + \frac{df_1}{d\tau}\Big]. \tag{3.15}$$

Solving the problem in region 2, which contains the products of combustion, we cannot assume potential flow—the gas entering this region through the flame front brings in vorticity perturbations. In this region it is convenient to use Landau's method of solution.

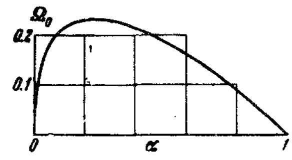

Fig. 3

Taking the divergence of both sides of (1.1), we find that the pressure perturbation in region 2 satisfies Laplace's equation. Thus, by analogy with our solution of (3.8), we take one term of the series

$$\frac{p_2'}{\alpha\rho_1 v_2^2} = \Phi_{mn}^{1}(\tau)\, \xi^n \sin m\varphi P_n^m(\cos\theta) \tag{3.16}$$

and assume, for convenience, that the arbitrary function of time $\Phi_{mn}^1(\tau)$ is related to another arbitrary function $f_2(\tau)$ by the relation

$$\Phi_{mn}^{1}(\tau) = \frac{df_2}{d\tau} - (n-1) f_2. \tag{3.17}$$

To determine the components of the perturbed velocity in region 2 we solve (1.1), taking into account (3.16),

$$v_{\xi 2}' / v_2 = n \sin m\varphi P_n^m [(n+1) f_3(\xi e^{\tau}) + f_2(\tau)\xi^{n-1}], \tag{3.18}$$

$$v_{\theta 2}' / v_2 = \sin m\varphi \frac{dP_n^m}{d\theta} [f_{3,\theta}(\xi e^{\tau}) + f_2(\tau)\xi^{n-1}], \tag{3.19}$$

$$v_{\varphi 2}' / v_2 = m \cos m\varphi P_n^m [f_{3,\varphi}(\xi e^{\tau}) + f_2(\tau)\xi^{n-1}] / \sin\theta. \tag{3.20}$$

Here f_3, $f_{3,\theta}$, $f_{3,\varphi}$ are arbitrary functions of the argument ξe^{τ}. These functions are related to each other, since the components of the velocity perturbation must satisfy the continuity condition (1.2). Substituting (3.18)–(3.20) into (1.2), we find that

$$f_{3,\theta} = f_{3,\varphi},$$
$$f_{3,\theta} = z\, df_3 / dz + 2f_3 = \partial f_3 / \partial\tau + 2f_3, \quad z = \xi e^{\tau}. \tag{3.21}$$

Thus, the velocity perturbations are

$$v_{\xi 2}' / v_2 = n \sin m\varphi [(n+1) f_3 + f_2\xi^{n-1}] P_n^m, \tag{3.22}$$

$$v_{\theta 2}' / v_2 = \sin m\varphi [f_2\xi^{n-1} + \partial f_3/\partial\tau + 2f_3]\, dP_n^m / d\theta, \tag{3.23}$$

$$v_{\varphi 2}' / v_2 = m \cos m\varphi [f_2\xi^{n-1} + \partial f_3/\partial\tau + 2f_3]\, P_n^m / \sin\theta. \tag{3.24}$$

Finally, we define the flame front perturbation

$$\zeta = \frac{\varepsilon}{v_2 t} = f_4(\tau) \sin m\varphi P_n^m(\cos\theta). \tag{3.25}$$

To determine the functions f_1, f_2, f_3, f_4 we use the boundary conditions (3.2)–(3.6) (with the variables r, θ, φ, t replaced by ξ, θ, φ, τ). Substituting the expressions for the velocity and pressure perturbations in regions 1 and 2, and the expression for the flame front perturbation into these boundary conditions, we find the following system of equations with constant coefficients:

$$(n+1)f_1 + \alpha n f_2 + \alpha n(n+1)f_3 + 3(1-\alpha)f_4 + (1-\alpha)\frac{df_4}{d\tau} = 0, \quad (3.26)$$

$$[1+\alpha(n+1)]f_1 + \frac{df_1}{d\tau} + \alpha(n-1)f_2 - \alpha\frac{df_2}{d\tau} + 2\alpha(1-\alpha)f_4 = 0, \quad (3.27)$$

$$f_1 - f_2 - 2f_3 - \frac{df_3}{d\tau} + (1-\alpha)f_4 = 0, \quad (3.28)$$

$$(n+1)f_1 + (3-2\alpha)f_4 + \frac{df_4}{d\tau} = 0. \quad (3.29)$$

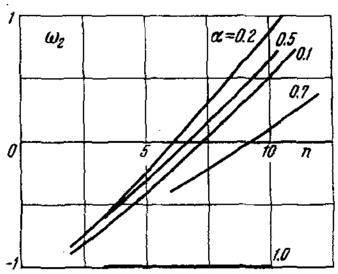

Fig. 4

In the following we shall investigate the time dependence of the flame front perturbation. Thus, instead of directly solving the system (3.26)–(3.29), we obtain an equation for the function f_4 and relations which connect all other functions with f_4 and with its derivative $df_4/d\tau$. The same procedure will be used later in the stability analysis with respect to higher spherical harmonics (§4), where we shall obtain a similar system of differential equations, with the difference that the coefficients will be functions of τ.

Adding the derivative of (3.26) to (3.28) multiplied by n(n − 1), and also subtracting the same derivative from (3.28) multiplied by (n − 1), we obtain

$$\alpha n(n-1)f_1 + (n+1)\frac{df_1}{d\tau} + \alpha n\frac{df_2}{d\tau} - \alpha n(n-1)f_2 + 2\alpha n\frac{df_3}{d\tau} - 2\alpha n(n-1)f_3 + \alpha n(n-1)(1-\alpha)f_4 + 3(1-\alpha)\frac{df_4}{d\tau} + (1-\alpha)\frac{d^2f_4}{d\tau^2} = 0, \quad (3.30)$$

$$(n^2-1)f_1 - (n+1)\frac{df_1}{d\tau} - \alpha n\frac{df_2}{d\tau} + \alpha n(n-1)f_2 - \alpha n(n+1)\frac{df_3}{d\tau} + \alpha n(n^2-1)f_3 + 3(1-\alpha)(n-1)f_4 + (1-\alpha)(n-4)\frac{df_4}{d\tau} - (1-\alpha)\frac{d^2f_4}{d\tau^2} = 0. \quad (3.31)$$

Equations (3.27), (3.29)–(3.31), together with the equation obtained by differentiating (3.29), constitute a system of five inhomogeneous linear algebraic equations for

$$f_1,\ \frac{df_1}{d\tau},\ F_2 = \frac{df_2}{d\tau} - (n-1)f_2,\ F_3 = \frac{df_3}{d\tau} - (n-1)f_3,\ \frac{d^2f_4}{d\tau^2}.$$

Each of these variables can be expressed as a linear function of f_4 and $df_4/d\tau$. After some calculations we obtain

$$\frac{d^2f_4}{d\tau^2} + a\frac{df_4}{d\tau} + bf_4 = 0, \quad (3.32)$$

$$a = \frac{2\alpha n^2 + 4n + 3\alpha n + 3\alpha}{n + \alpha n + \alpha},$$

$$b = \frac{-\alpha(1-\alpha)n^3 + 2\alpha n^2 + 3n + 3\alpha n - \alpha^2 n + 2\alpha}{n + \alpha n + \alpha}. \quad (3.33)$$

The general solution of (3.32) is

$$f_4 = C_1 t^{\omega_1} + C_2 t^{\omega_2}, \quad (3.34)$$

where ω_1 and ω_2 are the roots of the characteristic equation

$$\omega^2 + a\omega + b = 0, \quad (3.35)$$

and C_1, C_2 are constants of integration.

One of the roots of (3.35), ω_1 (the one with the minus sign in front of the square root), is always negative and does not represent instability. (In addition to that, the solution corresponding to this root is not bounded for $r \to 0$ and, therefore, $C_1 = 0$.) The other root of (3.35), ω_2 (the one with the plus sign), can take on different values depending on the particular values of n and α considered, and its sign is determined by the sign of the free term in (3.35). For a given α and sufficiently small n (but always n > 1) the root of the equation is negative (the flame is stable), and for large n the root is positive (the flame is unstable). The stability limit, which can be found by equating b to zero, is shown in Fig. 2 (curve 1).

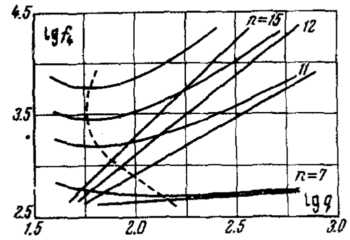

Fig. 5

The values of n for which the flame is stable lie below the curve. It can be seen that as α approaches either zero or unity the region of stability increases without bounds.

This is due to the fact that different degrees of thermal expansion during combustion correspond to different rates of growth of the perturbations. Figure 3 shows the rate of growth of the perturbations for a plane flame as a function of α. At the initial time this function coincides with the function representing the dependence of the characteristic frequency on α. (The figure shows the variable $\Omega_0 = -i\Omega_* \alpha / u_n k$, where Ω_* is the characteristic frequency according to Landau's theory [1]. The variable Ω_0 represents the distance, relative to the combustion products, over which the flame must move in order that the perturbation amplitude would change by a factor of e. The same variable determines the onset of instability in the case of a spherical flame.) It can be seen that Ω_0 vanishes at $\alpha = 0$ and at $\alpha = 1$. The maximum value

$$\Omega_{0m} = \sqrt{5} - 2$$

corresponds to

$$\alpha_m = \sqrt{5} - 2.$$

In accordance with Fig. 3, curve 1 of Fig. 2 has a minimum at a value of α which is close to α_m.

The nature of the time change of the perturbation amplitude of different spherical harmonics is shown also in Fig. 4, which represents the function $\omega_2(n)$ for various α. The points of intersection of the curves with the axis of abscissas determine the critical values of the harmonics.

Thus, the solution of the problem of the stability of propagation of a spherical flame in Landau's formulation shows that instabilities occur only in the case of perturbations with sufficiently large n, i.e., perturbations with short wavelengths, which, according to Landau's solution for the plane flame, have the highest rate of growth. However, it is precisely these short-wave perturbations for which Landau's approach ceases to be valid, since in the case of these perturbations one must take into account the effect of the transfer processes (viscosity, diffusion, thermal conductivity) and of the chemical kinetics, which may result in a damping of the perturbations.

§4. Stability analysis with respect to high spherical harmonics. In order to account for the effect of chemical kinetics and transfer processes on the stability of the flame in the case of perturbations corresponding to high spherical harmonics, when the wavelength is of the order of the flame front thickness, we shall use Markstein's assumption [3]. Markstein's assumption was that the over-all effect of transfer processes and chemical kinetics on the flame can be represented by a constant which relates the variation of the normal speed of the flame with its curvature:

$$u = u_n\left(1 + \frac{\mu}{\Lambda}\right) \tag{4.1}$$

(u and u_n are the perturbed and unperturbed flame speeds, Λ is the radius of curvature of the flame front, and μ is a constant).

It should be noted that Markstein's approach is, essentially, the next approximation to Landau's solution, and this approximation is linear. Equation (4.1) represents a series expansion of the flame speed in terms of a small parameter μ/Λ, which takes into account the structure of the flame. From dimensional considerations, the constant μ, which has the dimension of length, must be proportional to the flame front thickness, i.e., to $\mu = \mu_0 \varkappa / u_n$, where $\varkappa$ is the thermal diffusivity of the gas and μ_0 is a dimensionless factor, which was left undetermined in [3]. From general considerations it is clear that this factor must be a function of the Prandtl and Lewis numbers, the thermal expansion parameter α, and a dimensionless parameter which characterizes the temperature dependence of the rate of the chemical reaction. (In a first approximation it may be assumed that for large activation energies the rate of the chemical reaction in the flame depends only on the temperature in the reaction zone.) In [10] this factor was calculated on the basis of the thermal-diffusion formulation of the problem, disregarding the hydrodynamics. The formula obtained for μ_0 contained the large number $E/2RT_b$ (E is the activation energy of the chemical reaction, T_b is the combustion temperature, and R is the gas constant); therefore the factor μ_0 can reach values of the order 10–20.

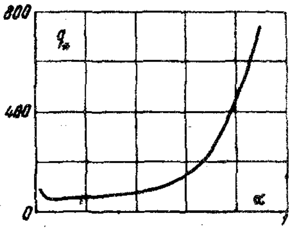

Fig. 6

The condition for the applicability of the linear approximation $\mu/\Lambda \ll 1$ imposes a restriction on the minimum perturbation wavelength for which the present formulation is meaningful. One cannot consider perturbation wavelengths which are too small (n too high), corresponding to such large curvatures that $\mu/\Lambda > 1$. (Note, that due to the increase of the wavelength corresponding to a given spherical harmonic, at large values of time we may consider large values of n.) Moreover, we cannot consider very low values of the radius R, such that $\mu/R > 1$, as in that case the speed of propagation of the unperturbed front is not constant.

The first restriction makes it impossible to determine the wavelength for which perturbations decay instead of growing. The linear correction cannot be used to find the stability limit. If we regard Landau's solution as the leading term of an asymptotic series for a perturbation Reynolds number tending to infinity, then the introduction of correction of the order $\mu/\Lambda \sim 1/N_{Re}$ should yield a correction for the frequency Ω, which characterizes the time variation of the perturbations, of the form

$$\Omega = \Omega_0\left[1 - \frac{F(\alpha)}{N_{Re}}\right]. \tag{4.2}$$

Thus, Markstein's solution consists in determining the specific form of the function $F(\alpha)$. Incidentally, this point has not been mentioned in [3], so that the expression for the frequency given there contains not only terms linear in μ/Λ, but also higher order terms. Carrying out the calculations, one obtains the formula for $F(\alpha)$

$$F(\alpha) = (\sqrt{\alpha + \alpha^2 - \alpha^3} + 1)/(1 + \alpha - \alpha^2 - \sqrt{\alpha + \alpha^2 - \alpha^3}). \tag{4.3}$$

The solution for the plane flame (4.2) shows a tendency for a variation of the characteristic frequency with decreasing perturbation wavelength. In order to make the frequency change sign and, consequently, to obtain a damping of the perturbations, one must extrapolate (4.2) to the range of wavelengths where $\mu/\Lambda \approx 1$. The determination of the stability limit by extrapolation of a linear correction constitutes a difficulty which is common to all existing studies of the problem.

The second restriction—the independence of the speed of propagation of the unperturbed flame of the flame front curvature—imposes a restriction on the minimum time for which the present formulation becomes meaningful. Taking into account that the curvature of a spherical surface is 2/R, we obtain from the condition $2\mu/R < \eta \ll 1$ the bound $v_2 t/\mu > 2/\eta$.

To carry out the stability analysis of the present problem we must change the boundary conditions (3.2)-(3.6): instead of (3.3) we must use the condition

$$p_1' + \frac{\partial p_1^\circ}{\partial r}\varepsilon = p_2' + 2\rho_1 v_2^2 \alpha (1-\alpha)\frac{\mu}{\Lambda}, \tag{4.4}$$

and instead of (3.6) we must use

$$v_{1r}' + \frac{\partial v_{r1}^\circ}{\partial r}\varepsilon - \frac{\partial \varepsilon}{\partial t} = -\alpha v_2 \frac{\mu}{\Lambda}. \qquad (4.5)$$

Since we are interested here in large n, for which the spherical flame is unstable according to Landau's formulation (n > 6, Fig. 2), we express the radius of curvature Λ of the spherical harmonic in the form

$$\frac{1}{\Lambda} = \frac{\varepsilon n(n+1)}{R^2} = \frac{\zeta n(n+1)}{v_2 t}. \qquad (4.6)$$

Taking account of the boundary conditions (3.2), (4.4), (3.4), (3.5), (4.5), and (3.7), we can obtain a system of equations for the functions f_1, f_2, f_3, f_4 of the same form as (3.26)–(3.29). Only some of the coefficients will be functions of time: the coefficient of f_4 in (3.27) will have an additional term $2\alpha(1 - \alpha)\mu/\Lambda$, and the coefficient of f_4 in (3.29) will have an additional term $-\alpha\mu/\Lambda$. Reducing the system to a single equation by the method used in §3, and taking into account that $\partial(\mu/\Lambda)/\partial\tau = -(\mu/\Lambda)$, we obtain the equation

$$\frac{d^2 f_4}{d\tau^2} + \left(a + \frac{\mu a_0}{v_2}e^{-\tau}\right)\frac{df_4}{d\tau} + \left(b + \frac{\mu b_0}{v_2}e^{-\tau}\right)f_4 = 0, \qquad (4.7)$$

where

$$a_0 = \alpha n(n+1)(2n+1)/(n + \alpha n + \alpha),$$
$$b_0 = a_0(n+1). \qquad (4.8)$$

Although n is now considered to be large, we cannot introduce any simplifications in the coefficient of (4.7), since the variable α can be small ($n\alpha \sim 1$). Moreover, we consider the last coefficient in the equation near those values of n and α for which the coefficient vanishes.

We seek a solution of (4.7) in the form of a sum of two terms. One term is the particular solution obtained without taking into account the effect of the curvature of the flame front on the speed, as in Landau's formulation. The second term is a correction which takes account of this effect, in the same way as in the plane flame case [cf. (4.2)]. To find the correction term it is convenient to introduce the new function U, which reduces (4.7) to the form

$$\frac{dU}{d\tau} + U^2 + \left(a + \frac{\mu a_0}{v_2}e^{-\tau}\right)U + b + \frac{\mu b_0}{v_2}e^{-\tau} = 0$$
$$\left(U = \frac{1}{f_4}\frac{df_4}{d\tau}\right). \qquad (4.9)$$

The solution of the problem in Landau's approximation is

$$\mu = 0, \; U = \frac{1}{f_4}\frac{df_4}{d\tau} = \omega = \text{const}.$$

(From here on we suppress the subscript on ω.) The solution we now seek is

$$U = \omega[1 + \chi(\tau)]. \qquad (4.10)$$

(Here $\chi(\tau)$ is a small correction factor.)

Substituting (4.10) into (4.9) and neglecting quadratic terms, we obtain

$$\frac{d\chi}{d\tau} + (2\omega + a)\chi + \frac{(\omega a_0 + b_0)\mu}{\omega v_2}e^{-\tau} = 0. \qquad (4.11)$$

The solution of (4.11) is

$$\chi = K\exp[-(2\omega + a)\tau] - \frac{\mu(\omega a_0 + b_0)}{v_2\omega(2\omega + a - 1)}e^{-\tau} \quad (4.12)$$

where K is an arbitrary constant. The condition $\mu = 0$, $\chi = 0$ yields K = 0. Thus, the solution of the problem in Markstein's approximation is

$$f_4 = \text{const}\left(\frac{v_2 t}{\mu}\right)^\omega \exp\frac{\mu(\omega a_0 + b_0)}{v_2 t(2\omega + a - 1)}. \qquad (4.13)$$

It can be easily seen that the dependence of f_4 on t for $\omega > 0$ (i.e., for the case when the solution according to Landau's approximation predicts instability) has a minimum at

$$\frac{v_2 t}{\mu} = \frac{a_0(\omega + n + 1)}{\omega(2\omega + a - 1)}. \qquad (4.14)$$

Such a dependence could be expected on physical grounds. At small values of t the wavelength of the perturbations on the surface of the flame is small, and the stabilizing influence of the curvature on the speed of propagation is effective. Therefore the initial perturbations decay. When the dimensions of the flame increase (for large values of t) the dimensions of the perturbations also increase, yielding the instability predicted by Landau—the perturbations grow with time.

For different n the minimum value of the relative amplitude corresponds to different times. However, for a given α there exists a value of the order of the spherical harmonic for which the minimum is reached faster than for other orders ($n = n_*$). The value of n_* can be easily found from the graphical representation of (4.14).

Figure 5 represents the relative amplitude of the perturbations as a function of time for $\alpha = 0.2$ and various values of n in the coordinates lg f_4 and lg $(v_2 t/\mu)$ = lg q. The straight lines in Fig. 5 represent the solution based on Landau's approximation. The broken line goes through the minima of the curves. For the value of α used in this figure $n_* = 12$.

The function $n_*(\alpha)$ is represented by curve 2 in Fig. 2. This curve lies inside the region bounded by curve 1.

Thus, there emerges the following picture of the development of the relative perturbations of the flame surface: At first perturbations of all wavelengths decay—the relative amplitude decreases. The different harmonics decay at different rates. After a certain time $q_* = v_2 t_*/\mu$, the amplitude of one of the harmonics ($n = n_*$) passes through a minimum and begins to grow. Subsequently the neighboring harmonics begin to grow, and an instability develops.

It is convenient to define the onset of instability of a spherical flame by means of the time q_*, which depends only on the thermal expansion parameter α. Figure 6 shows q_* as a function of α according to (4.14).

For most flames, for which $\alpha = 0.02-0.2$, the corresponding value is $q_* = 50-60$.

§5. **Discussion of results and comparison with experimental data.** The stability of propagation of a spherical flame was first investigated experimentally by Zel'dovich and Rozlovskii [5]. Subsequently, instability phenomena were investigated in more detail by Shchelkin and Troshchin and their collaborators [6–8].

In all these studies the onset of instability is represented by a critical value of the Reynolds number, based on the normal speed of the flame, the radius of the flame sphere, and the viscosity of the cold combustible mixture. It can be easily seen that the Reynolds number thus defined is proportional to a dimensionless time q:

$$N_{\mathrm{Re}}=\frac{u_n R}{\nu_1}=\frac{\mu_0}{P}q \text{ (P is the Prandtl number).} \quad (5.1)$$

In the experiments it is difficult to establish at what value of the flame radius one of the harmonics begins to grow. Usually one can observe only such states in which the perturbation of the surface is sufficiently developed. In [5], for instance, the onset of instability was defined by the transition from combustion to detonation, i.e., by a state of high turbulence. The corresponding values of the critical Reynolds number were high: 10^5-10^6. In [6–8] the spherical flame was observed photographically. The photographs were used to determine the time dependence of the absolute value of the surface perturbation. The onset of instability was defined as the time obtained by linear extrapolation of this dependence to zero perturbation. The critical Reynolds numbers determined by this method were smaller by one order of magnitude ($\sim 10^4$). However, in [6–8] the different rates of growth of different perturbations (different n) were neglected, which may lead to overestimated Reynolds numbers.

Regardless of the particular definition, the values of the critical Reynolds number obtained experimentally were in sharp disagreement with the predictions regarding the instability of a plane flame which were based on Landau's theory. These predictions disregarded, however, the specific characteristics of a spherical flame. The present results show that in the case of a spherical flame the value of the critical Reynolds number can be much higher than in the plane case. We shall illustrate this result by means of a numerical example.

Consider, for example, a flame with a thermal expansion parameter $\alpha = 0.2$. This corresponds to a critical time $q_* \approx 60$ (Fig. 6). Assuming $\mu_0 = 10$, $P = 1$, we obtain the result that the critical Reynolds number for the harmonic with $n = n_*$ is ≈ 600, i.e., two orders of magnitude larger than unity. Now let us assume that the Reynolds number determined experimentally is not this critical Reynolds number, but a Reynolds number corresponding to the time at which the perturbations at the flame front reach their initial values. Assume, for example, that the initial perturbations (caused by the spark, say) originated at $q = q_0 = 40$. (This value of q can be regarded as the initial time after which our original assumptions regarding the constancy of the speed of the unperturbed flame hold. For a viscosity $\nu_1 = 0.1$ cm²/sec and a flame speed $u_n = 100$ cm/sec, the flame radius at $q_0 = 40$ is $R_0 = 0.4$ cm, which is a reasonable value for a flame after ignition.) According to Fig. 5, the perturbation with $n = n_*$ will reach its initial value at $q \approx 100$, and the corresponding Reynolds number is ≈ 1430.

Although the numerical example demonstrates the considerable increase in the critical Reynolds number of a spherical flame as compared with a plane flame, due to the specific characteristics of the spherical flame, this is not sufficient to explain the experimental results in full. It can be assumed that the increase in the Reynolds number is also due to nonlinear stabilizing effects. One such effect has been discussed by Zel'dovich [11].

It is interesting to compare qualitatively the experimental data with the theoretical results. The photographs of the flames shown in [6–8] indicate that the instability of a spherical flame develops in the following way: First there appear large-scale perturbations, due to the perturbing effect of the spark, and these perturbations decay as the flame grows. After a certain time there appear perturbations on a much smaller scale than the initial ones, and these perturbations grow, reach considerably large dimensions, and form cellular flames, which eventually become turbulent. It should be noted that in the case of fast-burning flames the scale of the "secondary" perturbations is smaller, and these develop faster. The growth of the instability is strongly affected by the composition of the combustible mixture and, in particular, by the ratio between the thermal diffusivity and the mass diffusivity of the component which controls the combustion reaction. The onset of instability occurs sooner as the pressure increases.

These experimental results are in qualitative agreement with theoretical results. (For example, the effect of the composition of the combustible gas on the stability can be explained by a variation of the value of Markstein's constant [10], the effect of the pressure can be seen from the definition of the Reynolds number, the effect of the combustion temperature can be seen in Fig. 2, etc.)

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